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Topological Lagrangians and cohomology

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Abstract. *Witten* [12] *has interpreted the Donaldson invariants of four-manifolds by means of a topological Lagrangian. We show that this Lagrangian should be understood in terms of an infinite-dimensional analogue of the Gauss-Bonnet formula. Starting with a formula of Mathai and Quillen for the Thom class, we obtain a formula for the Euler class of a vector bundle, which formally yields the explicit form of Witten's Lagrangian. We use the same method to treat Lagrangians proposed for the Casson invariant.*

§1. INTRODUCTION

In [12] Witten introduced a Lagrangian leading to a topological quantum field theory in which the Donaldson invariants of 4-manifolds [7] appear as expectation values. That such a field theory should exist was suggested in [1], based on the work of Floer [8].

Witten's Lagrangian has been re-interpreted in the physics literature in a variety of ways [4, 9], but these are unlikely to help mathematicians understand the significance of Witten's work. In this paper we present an alternative approach which seeks to explain Witten's theory in terms of an infinite-dimensional version of the Gauss-Bonnet theorem (and its generalizations). In particular the explicit form of Witten's Lagrangian is derived as a direct consequence of standard

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formulas in differential geometry.

To understand how this comes about we recall that classical integration theory, dealing with the integral of scalar functions or measures, leads naturally to the exterior differential calculus when one considers integration on sub-manifolds. When systematized this leads to the de Rham theory connecting differential forms with homology. A similar story should hold, in appropriate cases, in infinite-dimensions. Now quantum field theories are formally described by a Feynman integral and it is therefore reasonable to expect differential forms to appear in such Feynman integrals, particularly in relation to topological questions. We shall show that this is indeed the case for Witten's theory and accounts for the Fermionic integrals that appear there.

In finite dimensions an oriented $2m$-dimensional real vector bundle E over a manifold X has an *Euler class* in $H^{2m}(X)$. In particular if $\dim X = 2m$, and X is compact and oriented we get an *Euler number* ϵ by evaluating the Euler class on X. This Euler number can also be interpreted as the number of zeros (counted with appropriate signs) of a generic section s of E. There are also integral formulas for ϵ of "Gauss-Bonnet" type i.e. expressions

$$\epsilon = \int_X \omega \tag{1.1}$$

where ω is a suitable de Rham representative of the Euler class of E. The classical Gauss-Bonnet theorem expresses ω in terms of the Pfaffian of the curvature of a connection on E. There is a more general formula, due to Mathai and Quillen [10] which depends on a connection and a section s. Their form is defined by

$$\omega_s = s^* U \tag{1.2}$$

where U is a closed differential form on E which is a "Gaussian representative" of the Thom class. This means it has Gaussian decay along each fibre E_x. Taking $s = 0$ gives the classical formula, while replacing s by ts with $t \to \infty$ and using stationary phase approximation gives the number of zeros of s (when generic).

When $n = \dim X > 2m$ the Mathai-Quillen form (1.2) is a $2m$-form on X. To get real numbers we have to form integrals

$$\int_X \eta \omega_s \tag{1.3}$$

where η runs over representatives of the $(n - 2m)$-dimensional cohomology of X.

In finite-dimensions the use of a non-zero section s to give the general formula (1.2), as opposed to the simple classical formula (1.1), is an unnecessary luxury. However in infinite dimensions it can be essential. It may be possible to make sense out of (1.2) when (1.1) is quite hopeless. It is then the Mathai-Quillen formalism comes into its own.

In fact we shall show that Witten's interpretation of the Donaldson invariants is precisely of the form (1.3) where X is the space of gauge equivalence classes of (irreducible) $SU(2)$-connections on a 4-manifold. The section s is the self-dual part of the curvature and the form ω_s is just the exponential of Witten's Lagrangian, while the forms η are the "observables" whose expectation value is being computed. The computation of (1.3) as

$$\lim_{t \to \infty} \int_X \eta \omega_{ts} \tag{1.4}$$

recovers Donaldson's definition of his invariants [7] in terms of the homology of the moduli space of instantons.

Since the appearance of [12] Witten has produced a number of other topological quantum field theories. At least one of these, the "Casson theory" can, as we shall now show, be explained in similar terms. In fact the Casson invariant of (homology 3-spheres) has been interpreted by Taubes [11] as the algebraic number of zeros of a cotangent vector field on the infinite dimensional manifold of (classes of) irreducible connections. A minor extension of the argument for the Donaldson invariant shows that this number can be written as an appropriate integral. This reproduces the Lagrangian which Witten has discussed. As with the Donaldson theory this uses a choice of Riemannian metric on the base manifold. However, unlike the Donaldson theory, where only the self-dual part of the curvature is used, the vector field of the Casson theory is the whole curvature and is metric independent. This means there is another Lagrangian (also due to Witten) for the Casson theory which is metric independent. Essentially this just uses delta functions at the zeros of the curvature and so is, in a sense more "singular". It can be derived as a limit of the metric - independent Lagrangian along the lines of (1.4).

There are three general comments that should be made at this stage. In the first place the symplectic or Liouville volume of a $2n$-dimensional symplectic manifold with symplectic form w is $w^n/n!$. Formally this can be replaced by $\exp(w)$, with the understanding that in performing the integration only differential forms in the top dimension (i.e. $2n$) give a non-zero answer. In infinite dimensions the exponential still makes sense but there is now no "highest dimension". This should be borne in mind when we write formulas with a view to an infinite-dimensional generalization.

The second point to note is that we shall be ignoring the singularities produced by reducible connections. This is tantamount, in finite dimensions, to ignoring boundary effects and is certainly not topologically justified in general. In fact the difficulties due to reducible connections are well-known both in the Donaldson and Casson theories. It is not yet clear to what extent these difficulties can be circumvented. Certainly Donaldson has shown, from his classical point of view, that his invariants are well-defined in considerable generality. We shall not however enter into these more subtle questions and we will content ourselves with a formal treatment consistent with Witten's work.

Finally it is perhaps worth pointing out that in deriving Witten's form of the Donaldson Lagrangian we nowhere need to use the fact that the space $\mathscr{A}$ of connections is an affine space, only that it is (formally) a Riemannian manifold. However, for the Casson invariant, explicit use has to be made of the affine structure of $\mathscr{A}$.

In §2 we shall describe in detail the formula of Mathai and Quillen [10] and apply this to the quotient of a Riemannian manifold by a compact group. This will then be applied formally in §3, with the manifold replaced by the affine space $\mathscr{A}$ of all G-connections on a compact oriented 4-manifold and the group replaced by the infinite-dimensional group $\mathscr{G}$ of all gauge transformations. We shall explicitly show how to derive Witten's Lagrangian [12].

In §4 and §5 we shall apply the same ideas to the Casson theory. We begin by extending our application of the Mathai-Quillen results to a slightly more general case. This is then applied to the space $\mathscr{A}$ of G-connections on a homology 3-sphere, and we derive the metric-dependent Lagrangian introduced by Witten [14]. Finally we show how this is related to the metric-independent Lagrangian.

We are greatly indebted to Simon Donaldson and Daniel Quillen who helped to provide the main ideas presented here. We are also grateful to Edward Witten for explaining to us all his ideas on topological quantum field theories.

§2. THE EQUIVARIANT EULER CLASS

We consider the following situation: G is a compact connected Lie group acting freely by isometries on an oriented Riemannian manifold P of dimension $2m + d$ (where $d = \dim G$) and V is a real vector space of dimension $2m$ with an orthogonal action of G. Thus $P \to P/G = X$ is a principal G-bundle and we can form the vector bundle E over X associated to the representation V. Using the Riemannian metric on P to define orthogonals to the G-orbits we get a connection for the principal bundle $P \to X$ and hence for the associated vector bundle E. The Euler form of E is then defined as a $(2m)$-form on X and we are interested in computing this in various ways. In particular we shall use a fixed G-invariant map $s : P \to V$ and the induced section of E.

Our starting point is the paper [10] by Mathai and Quillen. They define a certain explicit representative for the generating Thom class in $H_G^{2m}(V)$, the equivariant cohomology with compact supports of V. Their model of equivariant cohomology for any G-manifold W is the G-invariant elements of $\Omega^*(W) \otimes S(g^*)$ with the differential d_G defined as follows. $d_G = 0$ on elements of $S(g^*)$ and for $\phi \in \Omega^*(W)$ we define $d_G(\phi) \in \Omega^*(W) \otimes g^*$ by

$$d_G(\phi)_\xi = \mathrm{d}\phi - i_\xi(\phi) \quad \text{for} \quad \xi \in g.$$

The Mathai-Quillen element lies in $\Omega^*(V) \otimes S(g^*)$ and is defined by the formula

$$U = \pi^{-m} e^{-x^2} \int \exp\left(\frac{\chi^t \Omega \chi}{4} + i dx^t \chi\right) \mathscr{D}\chi \tag{2.1}$$

Here $x = (x_1, \ldots, x_{2m})$ are coordinates for $V = R^{2m}$ and $\chi = (\chi_1, \ldots, \chi_{2m})$ are Grassmann algebra variables. The Fermion integral $\int \mathscr{D}\chi$ means that we expand and take the coefficient of $\chi_1 \wedge \chi_2 \wedge \ldots \wedge \chi_{2m}$. Note that we get an even number of i and so a *real* answer. Ω is the skew-symmetric "universal curvature matrix" Ω_{jk}. More precisely it is the image of the G-curvature matrix under the representation

$$\rho : G \to SO(2m)$$

Remarks

(2.2) U does not have compact support but the exponential decay of the factor e^{-x^2} is equally good

(2.3) Mathai and Quillen write U in the alternative form

(2.4) $U = (2\pi)^{-m} Pf(\Omega)\, e^{-x^2 - dx^t \Omega^{-1} dx}$.

This can be viewed as an evaluation of the Fermionic integral (at the expense of inverting Ω). The equivalence of (2.1) and (2.4) is given in [10, (1.8)].

(2.5) The expressions $\chi^t\ dx$ and $\chi^t \Omega \chi$ in (2.1) clearly show that χ should be viwed, invariantly as a *tangent vector* to the manifold (here V), i.e. as an element of V itself. This also means that we can pass to the cotangent bundle T^*V with coordinates (x_i, ξ_i) and consider $\chi_i = d\xi_i$. Then $dx^t \chi = \Sigma dx_i \wedge d\xi_i$ is the canonical 2-form of the symplectic manifold T^*V, while $\chi^t \Omega \chi$ is a 4-form. The Fermionic integration can then be described in intrinsic geometric terms as follows: given a differential form on T^*V, restrict to the zero section, divide by the normal volume element and integrate.

(2.6) We should explain more about Ω. The representation ρ on the Lie algebra level assigns to each $\xi \in g$ a skew-symmetric matrix $\{\rho(\xi)_{jk}\}$ whose entries are linear forms in ξ. Hence ${}_j\Sigma_k\ \chi_j \rho(\xi)_{jk} \chi_k$ is a linear function on g with values in the exterior algebra of the χ_i. Thus the exponent in (2.1) is a linear function of ξ, dx.

If now $P \xrightarrow{G} X$ is a principal G-bundle with connection and curvature matrix Ω then we can interpret (2.1) as a G-equivariant form on $P \times V$. It represents the *Thom class* of the vector bundle $E = (P \times V)/G$. More precisely its *horizontal part* descends to a *closed form* on E which represents the Thom class. By this we mean the form U is to be evaluated on the projection of tangent vectors on the horizontal part of the tangent space to $P \times V$, with respect to the G-connection pulled back from the connection on P. In other words, if H_p is the horizontal subspace at p, then the horizontal subspace at (p, v) is $H_p \oplus V$.

Note. To get an explicit formula for this horizontal part we need to use the Weil algebra as in [10].

For a principal G-bundle $P \to X$ the connection is defined explicitly by a 1-form θ on P with values in g. The curvature is then

$$\Omega = d\theta + 1/2\,[\theta, \theta].$$

Note that (by definition) $\theta = 0$ on horizontal vectors so that Ω and $d\theta$ coincide on horizontal vectors. Thus in the formula for U we can replace Ω by $d\theta$ since we are only evaluating on horizontal vectors.

Now for any group action on a Riemannian manifold (not necessarily free) we have a canonical 1-form ν with values in g^* defined by

$$\nu_\xi(\alpha) = \langle C\xi, \alpha \rangle \qquad \xi \in g, \quad \alpha \text{ tangent vector}$$

where $C\xi$ is the tangent vector field defined by ξ and $\langle\ ,\ \rangle$ is the inner product of the metric. Clearly ν vanishes on horizontal vectors. To compare it with θ we need only evaluate it along G-orbits. Also we shall convert ν into a 1-form with values in g by using the Killing form on g. We then have an isomorphism

$$C : g \to T$$

where T is the tangent space along the G-orbit at a point of the manifold. Both spaces have inner products and these differ by $R = C^*C$, i.e.

$$\langle C\xi, C\eta \rangle = (R\xi, \eta)$$

where $\langle\ ,\ \rangle$ is the metric on T and $(\ ,\ )$ is the Killing form on g. Identifying g, with g^* by the Killing form, θ on vertical vectors is given by

$$\theta_\xi(\alpha) = (\xi, C^{-1}\alpha).$$

Putting $\beta = C^{-1}\alpha$ so that

$$\theta_\xi(\alpha) = (\xi, \beta)$$

while $\nu_\xi(\alpha) = \langle C\xi, C\beta\rangle = (R\xi, \beta)$ we see that $\nu = R\theta$ or

$$\theta = R^{-1}\nu. \tag{2.7}$$

Note that

$$d\theta = R^{-1}\,d\nu + d(R^{-1})\,\nu$$

and the last term vanishes on a pair of horizontal vectors. Thus, on horizontal vectors, we can replace $d\theta$ by $R^{-1}d\nu$. In particular we can do this for the Thom form (2.1).

In this form the explicit inverse R^{-1} is disagreeable. However, we may eliminate this by using the Fourier inversion formula. Recall that, for a single variable x, this takes the form

$$f(a) = \frac{1}{2\pi}\int e^{-ia\xi}\, e^{ix\xi}\, f(x)\, d\xi\, dx.$$

Replacing a by a/λ, ξ by $\lambda\xi$ we get

$$f(a/\lambda) = \frac{1}{2\pi}\int e^{-ia\xi} e^{ix\lambda\xi}\, f(x)\,\lambda\, d\xi\, dx.$$

The corresponding formula for d variables and λ replaced by a non-negative self-adjoint matrix R is then

$$f(R^{-1}a) = (2\pi)^{-d}\int e^{-i\langle a,\xi\rangle} e^{i\langle x, R\xi\rangle}\, f(x)\det R\; \mathscr{D}\xi\; \mathscr{D}x. \tag{2.8}$$

This shows that we can effect the substitution $x \to R^{-1}a$ by an integral formula that does not explicitly invert R.

We shall apply this with $n = \dim G = d$ and the Fourier transforms being over Lie algebra variables ϕ,λ. Using (2.8) and (2.1) with Ω replaced by $R^{-1}d\nu$ and $a = d\nu$ we get

$$U = (2\pi)^{-d}\pi^{-m}e^{-x^2}\int \exp\left\{\frac{\chi^t\rho(\phi)\chi}{4} + idx^t\chi - i\langle d\nu, \lambda\rangle + i\langle\phi, R\lambda\rangle\right\}\det R\; \mathscr{D}\chi\; \mathscr{D}\phi\; \mathscr{D}\lambda. \tag{2.9}$$

Recall that this formula is to be interpreted as giving a differential form on $E = (P \times V)/G$ by taking the *horizon ' part* on $P \times V$, descending to E. We can describe this process alternatively by first multiplying by the invariant volume form $\mathscr{D} g$ along the orbits, normalized so that

$$\int_G \mathscr{D} g = 1$$

and then integrating over the fibre. Now to get $\mathscr{D} g$ along the fibres we must use the connection form, and we can write this as a Fermionic integral over a Lie algebra variable η:

$$\mathscr{D}g = \int \exp(\theta, \eta) \mathscr{D}\eta \tag{2.10}$$

where (,) stands as before for the (normalized) Killing form. Replacing this by the metric on P we get

$$(\det R)\, \mathscr{D}g = \int \exp \langle \mathrm{d}A, C\eta \rangle\, \mathscr{D}\eta \tag{2.11}$$

where the A_i are coordinates on P. Substituting this into (2.9) eliminates the factor $\det R$ and leads to the formula

$$U = 2^{-d}\, \pi^{-d-m}\, e^{-x^2} \int \exp \left\{ \frac{\chi^t \rho(\phi) \chi}{4} + i dx^t \chi - i \langle \mathrm{d}\nu, \lambda \rangle \right. \tag{2.12}$$

$$\left. + i \langle \phi, R\lambda \rangle + \langle \mathrm{d}A, C\eta \rangle \right\} \mathscr{D}\eta\, \mathscr{D}\psi\, \mathscr{D}\phi \mathscr{D}\lambda.$$

Here in addition to the 2 Fermionic integrations over η, χ and the integrations over the Lie algebra variables ϕ, λ we also understand an integration over the fibre of $P \times V \to (P \times V)/G = E$.

This formula is our final formula for the *Thom class.* To get a formula for the *Euler class* of the vector bundle E over X we must pick a section $s : X \to E$ or equivalently a G-equivariant map $s : P \to V$ and pull back the Thom class.

Finally to get the Euler number of E (assuming X compact and oriented) we must integrate the Euler class over X. Replacing x by s in (2.12) we end up with the following *integral formula for the Euler number of E*

$$(2.13)\qquad 2^{-d}\pi^{-d-m}\int \exp\left\{-|s|^2 + \frac{\chi^t\rho(\phi)\chi}{4} + ids^t\chi - i\langle d\nu, \lambda\rangle \right.$$

$$\left. + i\langle\phi, R\lambda\rangle + \langle dA, C\eta\rangle\right\} \mathscr{D}\eta\, \mathscr{D}\chi\, \mathscr{D}\phi\, \mathscr{D}\lambda.$$

After integrating over η, χ, ϕ, λ we are left with a differential form on P and we then integrate this over P.

If we want to put (2.13) into the standard supersymmetric framework by eliminating differential forms we replace the dA_i by Fermionic ψ_i (representing a basis for the tangent space to P) and use the Fermionic integral formula

$$\int f(\psi)\, \mathscr{D}\psi\, \mathscr{D}A = f(dA).$$

Then (2.13) will become an integral of the form

$$(2.14)\qquad \int \exp(\mathscr{L})\, \mathscr{D}\psi\, \mathscr{D}\eta\, \mathscr{D}\chi\, \mathscr{D}A\, \mathscr{D}\phi\, \mathscr{D}\lambda$$

over 3 Fermionic and 3 Bosonic sets of variables.

§3. THE WITTEN LAGRANGIAN

We are going formally to apply formula (2.13), in the form (2.14), to the case when

$$G = \mathscr{G},\ P = \mathscr{A},\ V = \Omega^2_+(g),\ s = -F_+.$$

Here $\mathscr{A}$ is the space of irreducible connections on a principal G-bundle over a compact oriented 4-manifold M, $\mathscr{G}$ is the group of bundle automorphisms and $F_+ \in \Omega^2_+(g)$ is the self-dual part of the curvature. Thus $s = 0$ in $\mathscr{A}/\mathscr{G}$ gives the moduli space $\mathscr{M}$ of (anti-) instantons. We assume for the moment that we are in the case where we expect $\dim \mathscr{M} = 0$. Donaldson only treats the case $G = SU(2)$ because of subtleties involving singularities in the moduli space. Our treatment ignores such questions, so we work with general G.

We shall now identify the terms that occur in the Lagrangian $\mathscr{L}$ in (2.14). We have three Bosonic variables

$$A \in \Omega^1(M, g)$$

$$\lambda, \phi \in \Omega^0(M, g)$$

and three Fermionic variables

$$\psi \in \Omega^1(M, g)$$

$$\chi \in \Omega^2_+(M, g)$$
$$\eta \in \Omega^0(M, g)$$

To conform to Witten's notation, λ has been replaced by $\lambda/2$ throughout.

The inner product $(\, , \,)_i$ on $\Omega^i(M, g)$ is the Killing form combined with the usual inner product on $\Omega^i(M)$.

The individual terms are as follows:

1. $$-|s|^2 = -\int_M |F_+|^2 :$$

Since

$$\int_M (|F_+|^2 - |F_-|^2) = k$$

(the first Pontrjagin class), we may replace $-|s|^2$ by

$$-\frac{1}{2}\int_M |F|^2 - \frac{k}{2}.$$

Note that Witten eventually includes this topological factor in his Lagrangian: [12], (2.41).

2. The action of Lie($\mathcal{G}$) on $\mathcal{A}$ is given by $\phi \to d_A(\phi)$. Hence the operator R on Lie($\mathcal{G}$) is the Laplacian $\Delta_A = d_A^* d_A$. Thus the term $i/2\,(\phi, R\lambda)$ is

$$\frac{i}{2}\,(\phi, \Delta_A \lambda)_0.$$

3. $i\,\langle dA, C\eta\rangle$ is the l-form on $\mathcal{A}$ whose value on $\psi \in T_A\mathcal{A}$ is $i(\psi, d_A\eta)_I$.

4. $i\chi^t ds$: $ds(\psi) = -d_A^+\psi$, so this is a 1-form on $\mathcal{A}$ whose value on ψ is

$$-i(\chi, d_A\psi)_2.$$

5. $\chi^t\phi\chi/4 : \phi \in$ Lie($\mathcal{G}$) acts on χ by sending it to $[\chi, \phi]$. So the term is $1/4(\chi, [\chi, \phi])_2$. But since the Killing form is invariant under the adjoint action, this becomes (where Tr denotes the Killing form)

$$1/4 \int_M Tr([\chi, \chi]\, \phi).$$

6. $- i/2\ (d\nu, \lambda)$: ν is a one-form on $\mathscr{A}$ with values in Lie($\mathscr{G}$) given by

$$(\nu(\psi), \lambda)_0 = (\psi, d_A \lambda)_1 .$$

Holding one tangent vector ψ_1 fixed and varying A, $A \to \nu_A \psi_1$ is a function $\mathscr{A} \to$ Lie($\mathscr{G}$) whose derivative with respect to $\psi_2 \in T_A \mathscr{A}$ is given by

$$(\psi_2(\nu\psi_1), \lambda)_0 = (\psi_1, [\psi_2, \lambda])_1 = \int_M Tr(\psi_1 \wedge *[\psi_2, \lambda])$$

$$= \int_M Tr\,([\psi_1, *\psi_2]\, \lambda).$$

This expression is antisymmetric in ψ_1 and ψ_2. Since $\mathscr{A}$ is affine and ψ_1, ψ_2 are constant vector fields, their Lie bracket is 0. So our term is the 2-form given on ψ_1, ψ_2 by

$$- \frac{i}{2}\ (d\nu(\psi_1, \psi_2), \lambda) = \frac{i}{2} \int_M Tr\,([\psi_1, *\psi_2]\, \lambda).$$

To obtain Witten's Lagrangian we must replace the real variable ϕ in our formulas by $i\phi$ (corresponding to analytic continuation). We then obtain precisely the partition function for Witten's Lagrangian (2.7), with the addition of a topological term as in (2.41).

In the more general case dim $\mathscr{M} \neq 0$, one has an elliptic complex $\Omega^0 \xrightarrow{d_A} \Omega^1 \xrightarrow{d_A^+} \Omega^2_+$, whose index is $-$ dim $\mathscr{M}$ generically. This means formally that the Euler class we have described above has *codimension* dim $\mathscr{M}$, and to get a numerical invariant one must integrate it against other differential forms on $\mathscr{A}$. We recall Donaldson's construction of polynomial invariants in this case. If Q is the principal G-bundle from which the connections come, then $\mathscr{G}$ acts on Q so we may form the space $\mathscr{A} \times_{\mathscr{G}} Q := \mathscr{Q}$, which is a G-bundle over $\mathscr{A}/\mathscr{G} \times M$. Integrating $c_2(\mathscr{Q})$ over a class $\gamma_i \in H_{k_i}(M)$ and then restricting to $\mathscr{M} \subset \mathscr{A}/\mathscr{G}$ gives a map $\Phi : H_k(M) \to H^{4-k}(\mathscr{A}/\mathscr{G})$. When $(k_1, \ldots, k_r)$ satisfy

$$\sum_i (4 - k_i) = \dim \mathcal{M},$$

the Donaldson invariant is defined as:

$$\int_{\mathcal{M}} \Phi(\gamma_1) \wedge \ldots \wedge \Phi(\gamma_r).$$

If M has a metric, an explicit connection is given by the orthogonals to the G-orbits in $\mathcal{Q}$ for a certain metric on $\mathcal{Q}$: this comes from the metric on $\mathcal{A} \times Q$ given at (A, q) by the standard metric on $\mathcal{A}$ and the lifting of the M-metric to Q by means of the connection A and a metric on G. The curvature $\mathcal{F}$ can be calculated explicitly [2]: for $t_i \in T_q Q$, $\tau, \sigma \in T_A \mathcal{A}$ it is

$$\mathcal{F}_{2,0}(t_1, t_2) = F_A(t_1, t_2)$$
$$\mathcal{F}_{1,1}(t, \tau) = \tau(t)$$
$$\mathcal{F}_{0,2}(\tau, \sigma) = R^{-1}\, d\nu(\tau, \sigma)$$

(in our previous notation).

When $\dim \mathcal{M} \neq 0$, Witten formulates the Donaldson invariants as expectation values of certain operators (see [12] (3.40))

$$\int_{\gamma k} W_k,$$

where W_k is a k-form on M constructed out of the fields: he lists

$$W_0 = 1/2\, Tr\, \phi^2$$
$$W_1 = Tr(\phi \wedge \psi)$$
$$W_2 = Tr(1/2\, \psi \wedge \psi + i\phi \wedge F)$$
$$W_3 = iTr(\psi \wedge F)$$
$$W_4 = -\,1/2\, Tr(F \wedge F).$$

In fact these arise from the integral over $\mathcal{A}$ of the product of the above Euler class with classes $\Phi(\gamma)$ written explicitly in terms of the above $\mathcal{F}$. The components c_i^{4-i} of $c_2 = Tr(\mathcal{F}\mathcal{F})$ in $\Omega^i(M) \otimes \Omega^{4-i}(\mathcal{A})$ are given by [4]:

$$c_0^4 = Tr(\mathcal{F}_{0,2}\, \mathcal{F}_{0,2})$$
$$c_1^3 = 2\, Tr(\mathcal{F}_{0,2}\, \mathcal{F}_{1,1})$$

$$c_2^2 = Tr(\mathscr{F}_{1,1}\, \mathscr{F}_{1,1}) + 2\, Tr\, (\mathscr{F}_{2,0}\, \mathscr{F}_{0,2})$$
$$c_3^1 = 2\, Tr(\mathscr{F}_{2,0}\, \mathscr{F}_{1,1})$$
$$c_4^0 = Tr(\mathscr{F}_{2,0}\, \mathscr{F}_{2,0}) = Tr\, F_A^2\,.$$

Up to overall normalisation factors and the replacement of ϕ by $i\phi$, we find that the expectation value of Witten's operators

$$\int_{\gamma_i} W_{k_i}$$

[12, (3.40)] in the path integral is just our Euler class integrated against the corresponding forms

$$\int_{\gamma_i} c_{k_i}^{4-k_i} \in \Omega^{4-k_i}(\mathscr{A}).$$

(Recall that the double Fourier transform (2.8) turned any expression $f(R^{-1}\mathrm{d}\nu)$ into $f(\phi)$: this is what happens to $\mathscr{F}_{0,2}$).

§4. THE CASSON INVARIANT

We consider $H = SU(2)$ and Y a homology 3-sphere. From now on, $\mathscr{A}$ will denote connections on the bundle $Y \times H \to Y$, i.e. $\mathscr{A} = \Omega^1(Y, \mathfrak{h})$.

The Casson invariant was originally defined in terms of a Heegaard splitting of Y into $Y_1 \cup Y_2$ along a surface Σ, as the intersection number of the submanifolds of $Rep(\pi_1(\Sigma), H)$ coming from $\pi_1(Y_i)$. Since representations of $\pi_1(Y)$ correspond to flat connections on the trivial H-bundle over Y, one may equate this with the algebraic number of flat connections.

Taubes [11] has justified this, interpreting the Casson invariant as the "Euler number" of $\mathscr{A}/\mathscr{G}$ in the following way: the assignment $A \to *F_A$ defines a vector field on $\mathscr{A}$ which descends to a vector field on $\mathscr{A}/\mathscr{G}$. For a finite dimensional manifold X, the Euler number is the signed sum of the zeros of a generic vector field ν. The sign at a zero p is sign (det $\nabla\nu$) where $\nabla\nu : T_pX \to T_pX$ is the covariant derivative. In the infinite dimensional case, the zeros of $*F$ are the flat connections: their relative signs are defined via the spectral flow of a suitable operator $\nabla\nu$ along a path between two zeros. The absolute sign is defined via a technical procedure which relates everything to the product connection. To do this, Taubes switches from working on $\mathscr{A}/\mathscr{G}$ to working on $\mathscr{A} \times$ Lie $\mathscr{G}$. The signed sum of flat connections gives the Casson invariant.

Witten has proposed two Lagrangians whose partition functions represent the Casson invariant. The first of these [14] depends on the metric on Y, and arises from reduction of the Donaldson invariant Lagrangian from four to three dimensions: its Bosonic part is

$$(4.1) \qquad \mathscr{L}_B = \int_Y Tr(|F|^2 + |d_A\phi|^2 + |d_A\lambda|^2 + |d_A\sigma|^2$$
$$+ [\phi, \lambda]^2 + [\phi, \sigma]^2 + [\lambda, \sigma]^2)$$

where $\sigma \in \Omega^0(y, \ \mathfrak{h}\ )$ comes from the fourth component of the connection. (A similar dimensionally reduced Lagrangian is discussed in [3, 6]). The second Lagrangian [13] is independent of the metric on Y: it is

$$(4.2) \qquad \mathscr{L} = \int_Y Tr(B \wedge F_A + \chi \wedge d_A \psi)$$

where all fields are in $\Omega^1(Y, \mathscr{H})$ and (A, ψ) and (B, χ) are boson-fermion pairs. (This Lagrangian is discussed in [5]).

We derive an integral formula for the Casson invariant as an Euler number ϵ for $\mathscr{A}/\mathscr{G}$, similar to that described above for the Donaldson invariant.

§5. THOM AND EULER FORMS OF A QUOTIENT BUNDLE

In the Casson invariant situation, we wish to write the Euler number of the bundle $T(\mathscr{A}/\mathscr{G}) \to \mathscr{A}/\mathscr{G}$ as a formal integral over $\mathscr{A}$. This cannot be handled directly as the Donaldson invariant was, for this bundle is not associated to the principal bundle $\mathscr{A} \to \mathscr{A}/\mathscr{G}$. However, it is a *quotient* of two such associated bundles, and we shall see this leads to the required integral formula.

Let us establish some notation.

Suppose P is a principal G-bundle over X, and V, V' are even-dimensional inner product spaces on which G acts via representations into $SO(V)$, $SO(V')$. Suppose $P \times V'$ embeds in $P \times V$ by a G-map

$$(p, \sigma) \to \alpha(p, \sigma) = (p, \gamma(p)\, \sigma),$$

where $\gamma(p) : V' \to V$ is linear and injective. Thus the vector bundle $P \times_G V'$ embeds in $P \times_G V$: denote the resulting quotient bundle by E.

Let $s : P \to V$ be a G-map satisfying $s(p) \in (Im\ \gamma(p))^{\perp}$); this gives a section of E.

Define a G-map $\zeta : P \times V' \to P \times V$ by:

$$\zeta(p, \sigma) = (p, \gamma\,(p)\,\sigma + s(p)).$$

Our result is then given by the following

PROPOSITION: *With the above notation, the Euler class of E is $\pi'_*\,\zeta^*\omega$, where ω is the Thom class of $P \times_G V$ and $\pi' : P \times_G V' \to X$ is projection.*

This Proposition thus enables us to express the Euler class of E in terms of the Mathai-Quillen formula.

REMARK: In the case that interest us, $P = \mathcal{A}$, $X = \mathcal{A}/\mathcal{G}$ and $G = \mathcal{G}$, where $\mathcal{A}$ is the space of connections on the trivial $(SU(2) : = H)$-bundle over a homology 3-sphere Y. The spaces V and V' are $\Omega^1(Y, \mathfrak{h})$ and $\Omega^0(Y, \mathfrak{h})$, respectively the tangent space to the affine space $\mathcal{A}$ and the Lie algebra of $\mathcal{G}$. The group $\mathcal{G}$ acts on these by the adjoint action as in the Donaldson case. The map

$$\gamma(A) : \Omega^0(Y, \mathfrak{h}) \to \Omega^1(Y, \mathfrak{h})$$

is just d_A. One may verify $d_{A\cdot g}(g^{-1}\sigma g) = g^{-1}(d_A\sigma)g$ for $g \in G$, $\sigma \in \Omega^0(Y, \mathfrak{h})$ so this is indeed a $\mathcal{G}$-map. $(Im\,\gamma(A))^{\perp}$ is $ker\, d_A^*$. The section $s : \mathcal{A} \to \Omega^1(Y, \mathfrak{h})$ is $A \to {}^*F_A$; this is in $(Im\;\gamma(A))^{\perp}$ by the Bianchi identity.

Proof of Proposition: Let E_0 denote $P \times_G V$, and $E^{\perp}$ the image of $P \times_G V'$ under α. Let ω denote the Thom class of E_0. Since $E_0 = E \oplus E^{\perp}$, we have $\omega = uu^{\perp}$, where u and $u^{\perp}$ are the Thom classes of E and $E^{\perp}$. We observe that the bundle $E_0 \to E^{\perp}$ has a section $s \oplus id$ such that the following commutes:

$$\begin{tikzcd}
E \oplus E^{\perp} = E_0 \arrow[r, shift left] \arrow[d] & E^{\perp} \arrow[l, dashed, bend right, "s \oplus id"'] \arrow[d, "\pi"] \\
E \arrow[r, shift right] & X \arrow[l, dashed, bend left, "s"]
\end{tikzcd} \tag{5.1}$$

Thus the Euler class of E is

$$s^*u = \pi_*((s^*u)\,u^{\perp}) = \pi_*((s \oplus id)^*\omega).$$

Then consider

$$\begin{tikzcd}
P \times_G V' \arrow[rr, "\alpha"] \arrow[dr, "\pi'"'] & & E^{\perp} \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

and observe that $\zeta = (s \oplus id)\alpha$. Thus

$$\pi'_*(\zeta^*\omega) = \pi'_*(\alpha^*(s \oplus id)^*\omega)$$
$$= \pi'_* \alpha_*^{-1}(s \oplus id)^*\omega = \pi_*((s \oplus id)^*\omega),$$

as desired. This uses the fact that for a diffeomorphism α, the pushforward $\alpha_* = (\alpha^{-1})^*$. ■

Using the Mathai-Quillen formula (2.1) for ω, we can now write the integral of the Euler class over X as an integral over $P \times V'$ by using the volume form $\mathcal{D}g$ of G along the fibres of P ($\zeta^*\omega$ is basic since ζ is a G map). We obtain the following expression:

$$(5.2) \qquad \epsilon = \pi^{-m} \int_{P \times V'} \int \mathcal{D}\chi \exp\left\{ -|s(p)|^2 - |\gamma(p)\sigma|^2 + \frac{\chi^t \Omega \chi}{4} + i\chi^t(d[s(p) + \gamma(p)\sigma]) \right\} \mathcal{D}g.$$

As usual, we omit the terms in the exponential depending on the connection θ because of the factor $\mathcal{D}g$: d denotes exterior differentiation of a function on $P \times V'$. The Ω factor can be manipulated by a double Fourier transform and $\mathcal{D}g$ written as a Fermionic integral, exactly as for the Donaldson invariant.

In the particular case that interests us, we get formally

$$(5.3) \qquad \epsilon \sim \int_{\mathcal{A} \times \mathrm{Lie}(\mathcal{G})} \int \mathcal{D}\chi \exp\left\{ -|F_A|^2 - |d_A\sigma|^2 + \frac{\chi^t \Omega \chi}{4} + i\chi^t d[*F_A + d_A\sigma] \right\} \wedge \mathcal{D}g.$$

This looks very like the metric-dependent Lagrangian proposed by Witten for the Casson invariant. It has a scalar boson-fermion pair not present in the Donaldson invariant case, corresponding to the integral over Lie $\mathcal{G}$. The term $|d_A\sigma|^2$ is present in Witten's metric-dependent Lagrangian: his terms $|d_A\lambda|^2$, $|d_A\phi|^2$ result by substituting

$$\phi' = \phi - \lambda, \quad \lambda' = \phi + \lambda$$

into the term $(\phi, \Delta_A\lambda)$ that arises from the double Fourier transform.

Witten [13] also proposes a metric-independent Lagrangian for the Casson invariant. We shall see that this arises by taking a limiting case of the above formula and then evaluating the integral over V': the limiting procedure corresponds to the usual proof that the Euler class of a vector bundle is represented by the zero locus of a generic section.

In order to derive the metric-independent Lagrangian, we require an additional assumption:

(5.4) If $s(p)=0$, then $Im(\mathrm{d}s)_p \subset (Im\ \gamma(p))^{\perp}$.

In our case this is because $s(A) = {*d_A d_A}$ whereas $\mathrm{d}s_A = {*d_A}$.

We introduce the notation $\chi_T(p)$, $\chi_L(p)$ ("transverse" and "longitudinal") for those Fermionic variables in χ that form a basis for $(Im\ \gamma(p))^{\perp}$ (resp. $Im\ \gamma(p)$).

To obtain the limiting expression for ϵ, we first observe that the choices of s and γ were arbitrary – so we may replace them by ts, $t\gamma$ ($t \in \mathbb{R}$). This gives

$$(5.5)\qquad \epsilon = \pi^{-m}\int_{P\times V'}\int \mathscr{D}\chi \exp\{-t^2(|s(p)|^2+|\gamma(p)\,\sigma|^2)+ + \frac{\chi^t\Omega\chi}{4} + it\chi^t \mathrm{d}[s(p)+\gamma(p)\,\sigma]\}\wedge \mathscr{D}g.$$

Next we replace χ by $t\chi$ and take the limit as $t\to\infty$. $\mathscr{D}\chi$ becomes $t^{2m}\,\mathscr{D}\chi$, and the term $\chi^t\,\Omega\chi$ disappears in the limit.

Now for one real variable y, the distribution given by $\lim_{t\to\infty} te^{-t^2y^2}$ is just $\sqrt{\pi}\delta(y)$. So we get

$$(5.6)\qquad \epsilon = \int_{P\times V'}\int \mathscr{D}\chi_T\,\mathscr{D}\chi_L\,\delta_T(s(p))\,\delta_L(\gamma(p)\,\sigma)\times \times \exp i\{\chi_T^t[\mathrm{d}s_p(\psi)] + \chi_L^t\gamma(p)\,\tau\}\wedge \mathscr{D}g$$

where we introduce the notations ψ, τ for tangent vectors to P, V'. The integral over $\mathscr{D}\chi_L$ then yields

$$\det(\gamma(p))\,dvol(V')$$

and the integral

$$\int_{V'} \delta_L(\gamma(p)\sigma)\det(\gamma(p))\,dvol\,(V') = 1.$$

We are thus left with

(5.7) $$\epsilon = \int_P \mathscr{D}\chi_T \delta_T(s) \exp i(\chi_T^t \, ds) \wedge \mathscr{D} g.$$

This corresponds to Witten's metric independent Lagrangian since his factor

$$\int \mathscr{D}B \exp i \int_Y Tr(B \wedge F_A)$$

is just a way of writing $\delta(*F_A)$. The factor $\mathscr{D}g$ and the restriction of the Fermionic integral to $(Im\ \gamma(p))^{\perp}$ rather than all of V in (5.7) arises from gauge-fixing Witten's Lagrangian.

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